Duration : 55 minutes

EPFL

Analysis II Midterm

CGC/MT/MX/SV

Spring 2024

Answers

For the **multiple choice** questions, we give

$+3$ points if your answer is correct,

0 points if you give no answer or more than one,

-1 if your answer is incorrect.

For the **true/false** questions, we give

$+1$ points if your answer is correct,

0 points if you give no answer or more than one,

-1 points if your answer is incorrect.

Part I: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = e^{x-y} - x + e^{2y}\,.$$

Then

- [ ] the function f admits exactly one local maximum in $\mathbb{R}^2$
- [x] the function f admits exactly one local minimum in $\mathbb{R}^2$
- [ ] the function f admits a stationary point in $\mathbb{R}^2$ that is not a local extremum
- [ ] the function f does not have stationary points in $\mathbb{R}^2$

Question 2 : Let $(\boldsymbol{a}_n)$ be the sequence of elements in $\mathbb{R}^3$ defined by

$$\boldsymbol{a}_n = \left(\frac{(-1)^n}{n},\, (-1)^n,\, (-1)^n n\right)^{\mathrm{T}}, \qquad \text{for all } n \in \{1,2,3,\ldots\}.$$

Then

- [ ] the sequence is bounded
- [ ] the sequence has a converging subsequence
- [ ] the sequence has a bounded subsequence
- [x] the limit $\lim\limits_{n\to\infty} \boldsymbol{a}_n$ does not exist

Question 3 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

Then

- [ ] f is continuous at $(0,0)$, but $\dfrac{\partial f}{\partial x}(0,0)$ does not exist
- [ ] f is not continuous at $(0,0)$
- [x] f is continuous at $(0,0)$, $\dfrac{\partial f}{\partial x}(0,0)$ and $\dfrac{\partial f}{\partial y}(0,0)$ exist, but f is not differentiable at $(0,0)$
- [ ] f is differentiable at $(0,0)$

Question 4 : The solution $y(x)$ of the differential equation

$$x\,y'(x) = \frac{\big(y(x)\big)^2}{\ln(x)+1}$$

on the interval $(1,+\infty)$ with initial condition $y(e) = -\dfrac{1}{2\ln(2)}$ also satisfies

- [ ] $y(e^3) = -\dfrac{1}{2\ln(2)}$
- [ ] $y(e^3) = -\dfrac{1}{\ln(2)}$
- [x] $y(e^3) = -\dfrac{1}{3\ln(2)}$
- [ ] $y(e^3) = \dfrac{1}{4\ln(2)}$

Question 5: Let $f : \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}$ be the function defined by

$$f(x,y) = \frac{y^2 \sin^2(x)}{x^4 + y^2} \,.$$

Then

- [x] $\lim_{(x,y)\to(0,0)} f(x,y) = 0$
- [ ] $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist
- [ ] $\lim_{(x,y)\to(0,0)} f(x,y) = \frac{\pi^2}{4}$
- [ ] $\lim_{(x,y)\to(0,0)} f(x,y) = \frac{1}{2}$

Question 6: The solution $y(x)$ of the differential equation

$$y''(x) - 3y'(x) + 2y(x) = 2x^2 - 2x$$

with initial conditions $y(0) = 2$ and $y'(0) = 2$ also satisfies

- [x] $y(1) = 5$
- [ ] $y(1) = 5 + 2e - 2e^2$
- [ ] $y(1) = 5 - 2e^2$
- [ ] $y(1) = \frac{5}{4} + 2e$

Question 7: The subset

$$A = \{(x,y) \in \mathbb{R}^2 : x^2 \leq 1 \text{ and } y \leq x^2\} \subset \mathbb{R}^2$$

- [ ] is bounded and not closed
- [ ] is closed and bounded
- [ ] is not bounded and not closed
- [x] is closed and not bounded

Part II: true/false questions

For each question, mark the box (without erasing) TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 8 : Let A and B be two nonempty subsets of $\mathbb{R}^2$. Then the subset C defined by

$$C = \overline{A} \cup \overline{B}$$

is a closed subset of $\mathbb{R}^2$.

■ TRUE   □ FALSE

Question 9 : Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function and let $g : \mathbb{R}^2 \to \mathbb{R}$ be defined by $g(x, y) = \big(f(x, y)\big)^2$. If f is not differentiable at $(x, y) = (0, 0)$, then g is not differentiable at $(x, y) = (0, 0)$.

□ TRUE   ■ FALSE

Question 10 : Let D be a nonempty, open, and bounded subset of $\mathbb{R}^n$. If $f : D \to \mathbb{R}$ is a continuous function, then f attains its global maximum on D.

□ TRUE   ■ FALSE

Question 11 : Let $A = \mathbb{R}^3 \backslash \{(0, 0, 0)\}$. Then the boundary ∂A is the empty set.

□ TRUE   ■ FALSE

Question 12 : Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a function of class C^3. Then

$$\frac{\partial^3 f}{\partial x^2 \partial y}(x, y, z) = \frac{\partial^3 f}{\partial x \partial y \partial x}(x, y, z)\,, \qquad \text{for all } (x, y, z) \in \mathbb{R}^3.$$

■ TRUE   □ FALSE

Question 13 : Let $f : \mathbb{R} \to (0, 1)$ be a function such that $\lim_{t \to 0} f(t) = L > 0$, then

$$\lim_{(x,y)\to(0,0)} \frac{x^2 + y^2}{f\big(x^2 + y^2\big)} = 0\,.$$

■ TRUE   □ FALSE

Question 14 : If $f : \mathbb{R}^2 \to \mathbb{R}$ is a function of class C^2, then f admits a tangent plane at each point of its graph.

■ TRUE   □ FALSE